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Method for deriving rational solutions of some nonlinear evolution equations

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Abstract. Maps for nonlinear evolution equations are discussed. An algorithmic method for deriving rational solutions is presented. This approach is illustrated by examples which have solutions in the form of two truncated expansions.

1. Introduction

In this paper a method for deriving rational solutions of some nonlinear partial differential equations is presented. The approach is based on the use of those truncated expansions found by expanding a solution of the origin equation in a Laurent series.

The most important property of nonlinear evolution equations is the existence of rational solutions. They are interesting in themselves and especially are useful when solving problems containing many elements [1].

Rational solutions have been studied in a number of works. Airault, Mckean and Mozer [2] were the first to find rational solutions to the Korteveg–de Vries (KdV) equation. Later rational solutions to some nonlinear evolution equations were obtained by Ablowitz and Satsuma [3] and Adler and Moser [4, 5].

Rational solutions of many nonlinear evolution equations were also obtained by Weiss [6], who used the singular manifold method for their determination. However, he used only the one-singular manifold method in his approach and had to attract discrete symmetries to reach his solutions [7–10].

In this paper the two-singular manifold method [11, 12] for deriving rational solutions is used. This approach is applicable if the nonlinear evolution equations have two (and more) families at the expansion of solutions in a Laurent series. This method can be used for deriving rational solutions of the 'modified' equations. However, by using the Miura transformations, it actually allows us to obtain solutions to many other nonlinear equations. The approach used in this paper is actually close to a method applied by Weiss; however, it is much easier and is algorithmic.

Current interest in the Painlevé property is known to stem from the observations made by Ablowitz and Segur [13] and Ablowitz *et al* [14, 15] that reductions of partial differential equations of the soliton type give rise to ordinary differential equations whose movable singularities are only poles. This circumstance has reduced them to a famous hypothesis about the Painlevé property. The conclusion drawn is that the nonlinear partial equation is solved by the inverse scattering transform if all reductions of the given equation to the ordinary differential equations lead to those equations that have the Painlevé property.

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However, the application of this hypothesis in practice runs into problems when checking all those reductions which have nonlinear partial differential equations. In this connection Weiss *et al* [6] have suggested modification of this hypothesis for nonlinear partial differential equations. They are carrying out research into solutions of nonlinear partial differential equations as expansions in the Laurent series with singularities made in some surface. Essentially they have included a new function which contains information about properties of the equation being investigated. This method has proved very effective during the study of many properties of nonlinear partial differential equations [8–12].

The application of the singular manifold method for research into the 'modified' nonlinear equations has appeared less effective and Musette and Conte [11] and Conte *et al* [12] presented further generalizations of the singular manifold method when two families of the expansion of solution in the Laurent series are discounted.

Later it will be shown that the two-singular manifold method quickly gives rational solutions to some nonlinear partial differential equations.

The outline of this work is as follows. In section 2 those relations connecting some nonlinear partial differential equations to their singular manifold equations are considered. These relations illustrate that the solutions of some nonlinear equations can be obtained through solutions of the singular manifold equations using truncated expansions. The statement of this method is given in section 3. This is then used for the derivation of rational solutions to the fifth-order modified Korteveg–de Vries (MKdV) equation. Our approach for deriving rational solutions to the classical Boussinesq equation is applied in section 4.

2. Maps and integrability for nonlinear evolution equations

Let us show that relations exist between some nonlinear evolution equations and their singular manifold equations. These relations can be obtained using truncated expansions. To illustrate this let us use the MKdV equation.

It is well known that the solutions of the MKdV equation

$$E(u) = u_t - \tfrac{3}{2}u^2 u_x + u_{xxx} = 0 \tag{2.1}$$

can be presented in the form of two truncated expansions [6]

$$u = -\frac{2z_x}{z} + u_1 \qquad u_1 = \frac{z_{xx}}{z_x} \tag{2.2}$$

and

$$v = \frac{2\varphi_x}{\varphi} + v_1 \qquad v_1 = -\frac{\varphi_{xx}}{\varphi_x} \tag{2.3}$$

where $z(x, t)$ and $\varphi(x, t)$ are new functions that characterize singular manifolds.

It is also known that the singular manifold equation found during the Painlevé analysis of equation (2.1) takes the form [6]

$$D(z) = z_t + z_x\{z; x\} = 0 \tag{2.4}$$

where

$$\{z; x\} = \frac{z_{xxx}}{z_x} - \frac{3}{2}\frac{z_{xx}^2}{z_x^2} \tag{2.5}$$

is the Schwarzian derivative.

Recently the right relations were found that connect the MKdV and KdV hierarchies with their singular manifold equations [16, 17]. These relations, as applied to equations (2.1) and (2.4), take the forms

$$E(u) = A_z D(z) \qquad E(u_1) = B_z D(z) \tag{2.6}$$

$$E(v) = -A_\varphi D(\varphi) \qquad E(v_1) = -B_\varphi D(\varphi) \tag{2.7}$$

where operators A_z and B_z take the forms

$$A_z = \frac{\partial}{\partial x}\left(\frac{1}{z_x}\frac{\partial}{\partial x} - \frac{2}{z}\right) \qquad B_z = \frac{\partial}{\partial x}\left(\frac{1}{z_x}\frac{\partial}{\partial x}\right). \tag{2.8}$$

Subscripts in operators (2.8) mean variables in them. Equations $E(u_1) = 0$, $E(v) = 0$ and $E(v_1) = 0$ are equation (2.1) and $D(\varphi) = 0$ is equation (2.4).

From relations (2.6)–(2.8) the solutions to equation (2.1) can be obtained using truncated expansions (2.2) and (2.3) if the solutions to equation (2.4) are known.

Truncated expansions (2.2) and (2.3) also lead to two invariants [18, 19]

$$\omega_1 = u_x - \frac{u^2}{2} = u_{1x} - \frac{u_1^2}{2} = \{z; x\} \tag{2.9}$$

$$\omega_2 = v_x - \frac{v^2}{2} = v_{1x} - \frac{v_1^2}{2} = -\{\varphi; x\} \tag{2.10}$$

which are the Miura transformations for the KdV equation

$$Q(\omega) = \omega_t + 3\omega\omega_x + \omega_{xxx} = 0. \tag{2.11}$$

Taking into account equations (2.6) and (2.7) the following relations for the KdV equation can be written

$$Q(\omega_1) = L_z A_z D(z) \qquad Q(\omega_2) = -L_\varphi A_\varphi D(\varphi) \tag{2.12}$$

where the operator L_φ takes the form

$$L_\varphi = \left(\frac{\partial}{\partial x} + \frac{2\varphi_x}{\varphi} - \frac{\varphi_{xx}}{\varphi_x}\right) \tag{2.13}$$

which shows that the solutions to equation (2.11) can be obtained by the given solutions to equation (2.4) by formulae (2.9) and (2.10).

The above-mentioned maps are relative to the MKdV and the KdV equations which can be solved by inverse scattering transform. However, this raises the question of the connection of the existence of such maps and the integrability of the nonlinear equation. More than that there is also the good question of similar relations for nonintegrable evolution equations [20].

It is noted that examples of nonintegrable equations which have similar relations can be found but a different result ensues in these cases.

Let us discuss this. For example if we take the equation [21]

$$E_1(u) = u_t - 2u_x u_{xx} - 2uu_x^2 - u^2 u_{xx} + u_{xxxx} = 0 \tag{2.14}$$

then one can present the solutions of this equation in the form of two truncated expansions

$$u = -\frac{2z_x}{z} + u_1 \qquad u_1 = \frac{z_{xx}}{z_x} \tag{2.15}$$

and

$$v = \frac{3\varphi_x}{\varphi} + v_1 \qquad v_1 = -\frac{3}{2}\frac{\varphi_{xx}}{\varphi_x}. \tag{2.16}$$

What is more one can obtain the following maps taking into account formula (2.15)

$$E_1(u) = -A_z D_1(z) \qquad E_1(u_1) = B_z D_1(z) \tag{2.17}$$

where

$$D_1(z) = z_t + z_x \frac{\partial}{\partial x}\{z; x\} = 0. \tag{2.18}$$

However, it is not possible to use formula (2.16) to obtain relations like (2.17). In fact substituting (2.16) into equation (2.14) actually leads to some equation $D_n'(\varphi) = 0$ which does not coincide with equation (2.18). In this case it is impossible to suggest the relations for equation (2.14) and similarly for (2.17). There is no difficulty in understanding why the example was doomed to failure, the problem being that equation (2.14) is not solvable by the inverse scattering transform. It is significant that the Painlevé test for equation (2.14) shows that one family of the solution expansion passes it but another family does not and equation $D_n'(\varphi) = 0$ is not the singular manifold equation in this case.

Now one can formulate the sufficiency condition for the integrability of some nonlinear evolution equation.

Proposition 2.1. Let equation $E(u) = 0$ be given and its solution presented in the form of truncated expansions like (2.2) and (2.3) connecting the original equation with its singular manifold equations. Then equation $E(u) = 0$ can be solved when the singular manifold equation is an integrable equation.

Proof. The rigorous arguments of this proposition are not given. It must be kept in mind here that the singular manifold equation contains a number of properties. First, this equation is found at the Painlevé analysis of the original equation. Secondly, this equation is unique for every family of the solution expansion. Thirdly, every singular manifold equation has two symmetries such that if we use them for the one of the first family we will obtain the singular manifold equation of another family and vice versa. If the singular manifold equation has the above-listed properties it can be confirmed that the singular manifold equation is a solvable equation. Taking into account maps like (2.6) and (2.7) it can be seen that equation $E(u)$ is a solvable equation also. □

Remark 2.1. Two symmetries for the singular manifold equation are thought to lead to the integrability of this and in fact proposition 2.1 corresponds to the map of one solvable nonlinear equation into another solvable equation.

It is interesting to consider maps of the equation

$$G(u) = u_t + \frac{\partial}{\partial x}(u_{xxxx} + 5u_x u_{xx} - 5u^2 u_{xx} - 5uu_x^2 + u^5) = 0 \tag{2.19}$$

which were first written by Fordy and Gibbons [22]. This equation has four singular behaviours and the solutions of equations (2.19) can be presented in the form of the four truncated expansions, two of them are:

$$u = \frac{z_x}{z} + u_1 \qquad u_1 = -\frac{z_{xx}}{2z_x} \tag{2.20}$$

and

$$v = -\frac{2\varphi_x}{\varphi} + v_1 \qquad v_1 = \frac{\varphi_{xx}}{\varphi_x}. \tag{2.21}$$

These truncated expansions lead to the following two singular manifold equations [7]

$$D_1(z) = z_t + z_x[\{z; x\}_{xx} + \tfrac{1}{4}\{z; x\}^2] = 0 \tag{2.22}$$

and

$$D_2(\varphi) = \varphi_t + \varphi_x[\{\varphi; x\}_{xx} + 4\{\varphi; x\}^2] = 0. \tag{2.23}$$

However, the other two truncated expansions do not lead to concrete singular manifold equations because these expansions correspond to families at negative indices [23, 24].

Transformations (2.20) and (2.21) map the solutions of equations (2.22) and (2.23) into the solutions of equation (2.19) according to the following equalities

$$G(u) = -\tfrac{1}{2}A_z D_1(z) \qquad G(u_1) = -\tfrac{1}{2}B_z D_1(z) \tag{2.24}$$

$$G(v) = A_\varphi D_2(\varphi) \qquad G(v_1) = B_\varphi D_2(\varphi). \tag{2.25}$$

It is significant that there are the following invariants

$$\omega_1 = u_x + u^2 = u_{1x} + u_1^2 = -\tfrac{1}{2}\{z; x\} \tag{2.26}$$

$$\omega_2 = v_x - \frac{v^2}{2} = v_{1x} - \frac{v_1^2}{2} = \{\varphi; x\} \tag{2.27}$$

which are obtained from truncated expansions (2.20) and (2.21).

By applying the Miura transformations (2.26) and (2.27) to equation (2.19) one can find the relations

$$E_3(\omega_1) = -\tfrac{1}{2}L_z A_z D_1(z) \qquad E_3(\omega_1) = -\tfrac{1}{2}M_z B_z D_1(z) \tag{2.28}$$

$$E_4(\omega_2) = L_\varphi A_\varphi D_2(\varphi) \qquad E_4(\omega_2) = M_\varphi B_\varphi D_2(\varphi) \tag{2.29}$$

where operator M_z is the following

$$M_z = \frac{\partial}{\partial x} - \frac{z_{xx}}{z_x}$$

and $E_3(\omega_1) = 0$ and $E_4(\omega_2) = 0$ are the Caudrey–Dodd–Gibbon and Kaup–Kupershmidt equations [25–27].

$$E_3(\omega_1) = \omega_{1t} + 5\omega_1^2\omega_{1x} - 5\omega_{1x}\omega_{1xx} - 5\omega_1\omega_{1xxx} + \omega_{1xxxxx} = 0 \tag{2.30}$$

$$E_4(\omega_2) = \omega_{2t} + 10\omega_2\omega_{2xxx} + 25\omega_{2x}\omega_{2xx} + 20\omega_2^2\omega_{2x} + \omega_{2xxxxx} = 0. \tag{2.31}$$

Assurance is given that equations (2.30) and (2.31) have the solutions which are found by formulae (2.26) and (2.27) at given solutions of equations (2.22) and (2.23).

It is necessary to note that it will not be possible to find relations like (2.24), (2.25), (2.28) and (2.29) if we take the other two truncated expansions for solutions of equation (2.19). In this case these truncated expansions, as distinct from (2.20) and (2.21), do not correspond to families of solutions whose movable singularities are poles. These families have negative indices that can be investigated by special methods [23, 24]. However, equations (2.19), (2.30) and (2.31) can be solved by inverse scattering transform [25–27] and they possess the Painlevé test.

3. Method for deriving rational solutions

We have presented the relations which connect some nonlinear evolution equations with their singular manifold equations. These relations were obtained using the truncated expansions of solutions like (2.2) and (2.3).

Let us describe the algorithm which we are going to apply for deriving rational solutions of some nonlinear evolution equations. Let the original equation $E(u) = 0$ be given and let its rational solutions be found. Let us assume that the solution of this equation can be presented in the form of two truncated expansions

$$u = a\frac{z_x}{z} + u_1 \tag{3.1}$$

and

$$v = b\frac{\varphi_x}{\varphi} + v_1 \qquad b \neq a \tag{3.2}$$

where u_1 and v_1 depends on derivatives of z and φ.

First, singular manifold equations have to be found which correspond to the representations of solutions (3.1) and (3.2). Secondly, it is important to analyse these singular manifold equations as being subject to symmetry. Thirdly, the maps of the singular manifold equations can be placed into the original equation $E(u) = 0$.

Then the following equality should be assumed:

$$u = v_1 \tag{3.3}$$

or

$$v = u_1. \tag{3.4}$$

In the case when the singular manifold equations are different equations (3.3) and (3.4) can be used jointly.

Equalities (3.3) and (3.4) and singular manifold equations are applied to find the rational solutions for the given equation: $E(u) = 0$. Let this approach be demonstrated using the fifth-order MKdV equation.

This equation can be written in the form [7]

$$u_t + \frac{\partial}{\partial x}\left(u_{xxxx} - \frac{5}{2}u^2 u_{xx} - \frac{5}{2}u u_x^2 + \frac{3}{8}u^5\right) = 0. \tag{3.5}$$

Equation (3.5) has the four families of the solution expansion but we have to take the truncated expansions corresponding to the two principal families

$$u = -\frac{2\varphi_x}{\varphi} + u_1 \qquad u_1 = \frac{\varphi_{xx}}{\varphi_x} \tag{3.6}$$

and

$$v = \frac{2\Psi_x}{\Psi} + v_1 \qquad v_1 = -\frac{\Psi_{xx}}{\Psi_x}. \tag{3.7}$$

The singular manifold equations which correspond to equation (3.5) were found by Painlevé analysis of equation (3.5). They coincide for two families and take the form [7]

$$\varphi_t + \varphi_x[\{\varphi; x\}_{xx} + \tfrac{3}{2}\{\varphi; x\}^2] = 0. \tag{3.8}$$

Using truncated expansions (3.6) and (3.7) the four relations between equation (3.5) and its singular manifold equation (3.8) like (2.6)–(2.9) can be obtained. Consequently, solutions to equation (3.5) can be found as solutions to equation (3.8) taking into account the four formulae of (3.6) and (3.7).

Equating $u = v_1$ we have

$$\frac{2\varphi_x}{\varphi} - \frac{\varphi_{xx}}{\varphi_x} = \frac{\Psi_{xx}}{\Psi_x} \tag{3.9}$$

which gives the relation of Weiss [7]

$$\Psi_x = c(t)\frac{\varphi^2}{\varphi_x} \tag{3.10}$$

after integration over x.

Using equality $u_1 = v$ one can obtain formula (3.10) as well.

Assuming $\varphi = z_n$, $\Psi = z_{n+1}$ we have the iterative formula

$$z_{n+1,x} = \frac{z_n^2}{z_{nx}} \tag{3.11}$$

for deriving rational solutions of equations (3.5) and (3.8).

Let $z_0 = x$ then we obtain from equations (3.11) and (3.8)

$$\begin{aligned} z_1 &= x^3 \\ z_2 &= x^5 - 720t \\ z_3 &= x^7 - 5040x^2t - 1209\,600t^2x^{-3} \end{aligned} \tag{3.12}$$

and so on.

4. Rational solutions of the classical Boussinesq equation

The classical Boussinesq, or Broer–Kaup, system

$$u_t + \omega_x + uu_x = 0 \tag{4.1}$$

$$\omega_t + u_{xxx} + (u\omega)_x = 0 \tag{4.2}$$

is equivalent to the scalar partial differential equation [12]

$$E(u) = u_{xxx} - \tfrac{3}{2}u^2u_x - 2uu_t - u_x\partial^{-1}u_t - \partial^{-1}u_{tt} = 0. \tag{4.3}$$

Solutions of equation (4.3) can be presented in the form of two truncated expansions

$$u = -\frac{2\varphi_x}{\varphi} + u_1 \qquad u_1 = -\frac{\varphi_t}{\varphi_x} + \frac{\varphi_{xx}}{\partial_x} \tag{4.4}$$

and

$$v = \frac{2\Psi_x}{\Psi} + v_1 \qquad v_1 = -\frac{\Psi_t}{\Psi_x} - \frac{\Psi_{xx}}{\Psi_x}. \tag{4.5}$$

Equation (4.3) can be rewritten in the following forms [28, 29]

$$E(u) = \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + u + \partial^{-1}\frac{\partial}{\partial t}\right)\left(u_x - \frac{1}{2}u^2 - \partial^{-1}u_t\right)\right] = 0 \tag{4.6}$$

and

$$E(v) = \frac{\partial}{\partial x}\left[\left(-\frac{\partial}{\partial x} + v + \partial^{-1}\frac{\partial}{\partial t}\right)\left(-v_x - \frac{1}{2}v^2 - \partial^{-1}v_t\right)\right] = 0. \tag{4.7}$$

Using formulae (4.4) and (4.5) it can be seen that there are equalities

$$\omega_1 = u_x - \tfrac{1}{2}u^2 - \partial^{-1}u_t = u_{1x} - \tfrac{1}{2}u_1^2 - \partial^{-1}u_{1t} \tag{4.8}$$

and

$$\omega_2 = -v_x - \tfrac{1}{2}v^2 - \partial^{-1}v_t = -v_{1x} - \tfrac{1}{2}v_1^2 - \partial^{-1}v_{1t}. \tag{4.9}$$

These invariants lead to the singular manifold equations of (4.1) that take the forms [29]

$$S - \tfrac{1}{2}C^2 + 2C_x - \partial^{-1}C_t = 0 \tag{4.10}$$

and

$$S - \tfrac{1}{2}C^2 - 2C_x - \partial^{-1}C_t = 0 \tag{4.11}$$

where [18, 19]

$$S = \{\varphi; x\} \qquad C = -\frac{\varphi_t}{\varphi_x}. \tag{4.12}$$

Assuming $u_1 = v$ we have

$$\frac{\varphi_t - \varphi_{xx}}{\varphi_x} = \frac{\Psi_t + \Psi_{xx}}{\Psi_x} - \frac{2\Psi_x}{\Psi}. \tag{4.13}$$

One can organize the iterative process using $\Psi = z_n$ and $\varphi = z_{n+1}$ for deriving rational solutions of equation (4.3)

$$\frac{z_{n+1,t} - z_{n+1,xx}}{z_{n+1,x}} = \frac{z_{n,t} + z_{n,xx}}{z_{n,x}} - \frac{2z_{nx}}{z_n}. \tag{4.14}$$

If we take $z_0 = x$ then we obtain

$$\begin{aligned} z_1 &= x^2 - 2t \\ z_2 &= x^3 - 6xt \\ z_3 &= x^4 - 12x^2t + 12t^2 \\ z_4 &= x^5 - 20x^2t + 60xt^2 \end{aligned} \tag{4.15}$$

and so on.

Assuming $u = v_1$ other rational solutions of equation (4.3) can also be obtained.

5. Conclusion

Let us emphasize the results of this work. Using the two-singular manifold method [11, 12] we have presented the approach for deriving rational solutions of some nonlinear evolution equations. We have also discussed maps of the singular manifold equations into some nonlinear evolution equations. This allowed the proof of our method. Rational solutions to the fifth-order MKdV and the classical Boussinesq equations were used as examples to demonstrate the application of this method.

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